Problem 3

Let a, b, c be three distinct integers, and let P be a polynomial with integer coefficients. Show that in this case the conditions

$$P(a) = b, \quad P(b) = c, \quad P(c) = a,$$

cannot be satisfied simultaneously.

Solution by Julia Xin

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$. Then,

$$P(a) - P(b) = a_n(a^n - b^n) + a_{n-1}(a^{n-1} - b^{n-1}) + ... + a_1(a - b) = (a - b)x, \quad \text{for some } x \in \mathbb{Z}.$$

In a similar way, we get

$$P(b) - P(c) = (b - c)y, \quad \text{for some } y \in \mathbb{Z},$$

$$P(c) - P(a) = (c - a)z, \quad \text{for some } z \in \mathbb{Z},$$

Moreover,

$$P(a) - P(b) = b - c, \quad P(b) - P(c) = c - a, \quad P(c) - P(a) = a - b.$$

Then, we deduce that

$$b - c = (a - b)x, \quad c - a = (b - c)y, \quad a - b = (c - a)z.$$

This implies that $xyz = 1$. The only options are $x = y = z = 1$ which implies $a = b = c$. But this is not possible because we assumed that a, b, c were different. The other option is that one of the integers x, y, z is 1 while the other two are -1. But this also implies that $a = b = c$.